Absence of singular spectrum for a perturbation of a two-dimensional Laplace-Beltrami operator with periodic electromagnetic potential

1998 J. Phys. A: Math. Gen. 31 7593

(http://iopscience.iop.org/0305-4470/31/37/017)

J. Phys. A: Math. Gen. **31** (1998) 7593–7601. Printed in the UK PII: S0305-4470(98)89160-9

Absence of singular spectrum for a perturbation of a two-dimensional Laplace–Beltrami operator with periodic electromagnetic potential

Abderemane Morame†

UMR 6629 CNRS, Department of Mathematics, University of Nantes, 2 rue de la Houssiniere, BP 92208, F-44322 Nantes Cedex 3, France

Received 10 November 1997, in final form 18 May 1998

Abstract. Let Γ be a lattice on $\mathbb{R}^2$. We consider a metric g, a one-form A and a real function V on $\mathbb{R}^2$, all Γ periodic. We prove that the spectrum of the Schrödinger operator on $L^2(\mathbb{R}^2)$, $u \mapsto P_g(D-A)u + Vu = (\mathrm{id} + A)^\star(\mathrm{id}u + uA) + Vu$, is absolutely continuous.

1. Introduction

Let Γ be a lattice on $\mathbb{R}^2$, and $g = (g_{jk})$ be a C^∞, Γ-periodic metric on $\mathbb{R}^2$,

$$g_{x-a} = g_x \qquad \forall a \in \Gamma \qquad g_{jj} > 0 \text{ and } |g| = \det(g_{jk}) > 0. \tag{1.1}$$

We consider a real C^∞, Γ-periodic one-form A (a magnetic potential),

$$A = A_1(x)\,\mathrm{d}x_1 + A_2(x)\,\mathrm{d}x_2 \qquad A_j(x-a) = A_j(x) \qquad \forall a \in \Gamma \tag{1.2}$$

and a Γ-periodic electrical potential

$$V : \mathbb{R}^2 \mapsto \mathbb{R} \qquad V \in L^\infty(\mathbb{R}^2) \qquad V(x-a) = V(x) \qquad \forall a \in \Gamma. \tag{1.3}$$

Hence the Schrödinger operator $P_g(D-A) + V = (\mathrm{id} + A)^\star(\mathrm{id} + A) + V$,

$$P_g(D-A) = \sum_{1\leqslant j,k\leqslant 2} |g_x|^{-1/2}(D_{x_j} - A_j(x))g^{jk}(x)|g_x|^{1/2}(D_{x_k} - A_k(x)) \tag{1.4}$$

is self-adjoint on $L_g^2(\mathbb{R}^2) = L^2(\mathbb{R}^2; |g|^{1/2}\,\mathrm{d}x)$, $(\mathrm{d}x = \mathrm{d}x_1 \wedge \mathrm{d}x_2)$, with domain the Sobolev space of order two, $\mathcal{D}(P_g(D-A)+V) = H^2(\mathbb{R}^2)$. $(D_x := -\mathrm{i}\frac{\partial}{\partial x} = (D_{x_1}, D_{x_2}), D_{x_j} = -\mathrm{i}\partial_{x_j} = -\mathrm{i}\frac{\partial}{\partial x_j}.)$

We will identify the magnetic field $\tilde{B} = \mathrm{d}A$ with the real function

$$B(x) = \frac{\partial}{\partial x_1}A_2(x) - \frac{\partial}{\partial x_2}A_1(x) \qquad (\tilde{B} = B(x)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2). \tag{1.5}$$

Green points out in [G] some spectral differences between $P_g(D) + V$ and $P_{g_0}(D) + V$, with $g_0 = (\delta_{jk})$ the flat metric. Among other things, one can find in [G] examples of conformal metric $g = c_N g_0$ with more than N gaps in the spectrum of $P_{c_N g_0}(D) + V$.

Other such examples can be constructed easily for general metric and with particular large magnetic field, using the localization of the spectrum of $P_g(D - \lambda A) + V$, as an operator on $L_g^2(\mathbb{T}^2) = L^2(\mathbb{T}^2, |g|^{1/2}\,\mathrm{d}x)$, for large constant λ, established in [H-M-1].

† E-mail address: morame@math.univ-nantes.fr

We let $\mathbb{T}^2$ be the Γ-related torus, $\mathbb{T}^2 := \mathbb{R}^2/\Gamma$, equipped with the Lebesgue measure associated to the metric g, $|g|^{1/2}\,\mathrm{d}x$.

Let us remark that $P_{g_0}(D-A)+V$ has a finite number of gaps in its spectrum, see [Sk] for the case without magnetic field and [Mh] for the general case. A spectral question is important. Are there any eigenvalues for $P_g(D-A)+V$? For the flat metric the answer is no, as it is well known for when $A=0$, see [R-S], and recently proved for $P_{g_0}(D-A)+V$ in [B-S] (see also [So] for higher dimensions).

Theorem 1.1. Under the above assumptions, the spectrum of $P_g(D-A)+V$ is absolutely continuous.

It follows from this theorem that $P_g(D-A)+V$ has no eigenvalues and its spectrum is formed by intervals (with positive lengths).

The strategy of our proof is the Thomas one [T] performed first for $-\Delta+V$ in dimension three, and generalized for all dimensions in [R-S]. As in [B-S], the operator that we use is a perturbation of a product of two elliptic operators, but we differ by not requiring scalar operators. We consider the Schrödinger operator with spin $P_{g,s}(D-A) = P_g(D-A)\mathbf{1}_2 + |g|^{-1/2}B(x)\sigma_3$ as a product, $P_{g,s}(D-A) = \tilde{\mathcal{D}}_g(D-A)\mathcal{D}_g(D-A)$, with $\mathcal{D}_g(D-A)$ a perturbation of the Dirac operator. As in [K], using multidimensional analytic extension in the Thomas approach, we have just to prove that the operator on $(L^2(\mathbb{T}^2))^2$ defined by $P_{g,s}(D-A-\Theta)$ has no Θ-independent eigenvalue, for $\Theta = \theta_r\,\mathrm{d}x + \mathrm{i}\theta_i\,\mathrm{d}x$, $\theta_r, \theta_i \in \mathbb{R}^2$.

It is easy to find $e \in \mathbb{S}^1$ and $c>0$ such that $E(\mathcal{D}_g(D-A-\Theta))^\star\mathcal{D}_g(D-A-\Theta)E \geqslant c|\theta_i|^2E$, if E is the projection $Eu(x) = (u(x)\cdot e)e, \forall u(x) \in (C^\infty(\mathbb{T}^2))^2$.

Exploiting gauge invariance of the spectrum in the analytic extension in the direction ω, ($\theta_i = \lambda\omega$, $\lambda>0$), we hope to show that $\tilde{\mathcal{D}}_g(D-A-\Theta)$ has a uniformly bounded inverse. We can easily neglect the magnetic field thanks to the dimension, and so the main difficulty comes from the metric, but we can find a sequence $(\Theta_k)_k$, with non-bounded imaginary part, such that $((\tilde{\mathcal{D}}_g(D-A-\Theta_k))^{-1})_k$ is uniformly bounded on $(L^2(\mathbb{T}^2))^2$. Then we get that $E(P_{g,s}(D-A-\Theta_k))^\star P_{g,s}(D-A-\Theta_k)E \geqslant c|\theta_{i,k}|^2E$ and we can conclude.

We hope that our method can be applied for higher dimensions.

Let us remark that later on, we may take $\Gamma = \mathbb{Z}^2$ if we modify g consequently, using the invariance of the spectrum by the action of the linear group $GL(2;\mathbb{R})$.

2. Some basic facts from Floquet theory

Let $\Gamma^\star$ be the dual lattice, $\Gamma^\star = \{\gamma \in \mathbb{R}^2; \gamma a \in 2\pi\mathbb{Z}, \forall a \in \Gamma\}$.

$\mathbb{K}^\star$ will denote a basic period cell of the dual lattice, i.e. when $\Gamma = \mathbb{Z}^2$, $\mathbb{K}^\star = ([0,2\pi[)^2$.

$\mathbb{K}^\star$ will be equipped with the normalized Lebesgue measure $\mathrm{d}\tilde{\theta} = \frac{\mathrm{d}\theta}{|\mathbb{K}^\star|}$.

From now on, we will identify every $\Theta = (\Theta_1, \Theta_2) \in \mathbb{C}^2$ with the closed one-form $\Theta_1\,\mathrm{d}x_1 + \Theta_2\,\mathrm{d}x_2$.

We recall that Floquet theory, see [R-S, K], is valid for $P_g(D-A)+V$, so there exists a unitary operator U, $U : L^2_g(\mathbb{K}^\star \times \mathbb{T}^2) \mapsto L^2_g(\mathbb{R}^2)$, such that

$$U^{-1}(P_g(D-A)+V)U = P(D-A-\theta)+V = \int_{\mathbb{K}^\star}^{\oplus} P^\theta_{g,\mathbb{T}^2}\,\mathrm{d}\tilde{\theta}$$

$$P(D-A-\theta) = \sum_{1\leqslant j,k\leqslant 2} |g(x)|^{-1/2}(D_{x_j}-\theta_j-A_j(x))g^{jk}(x)|g(x)|^{1/2}(D_{x_k}-\theta_k-A_k(x)) \tag{2.1}$$

is to be considered as a self-adjoint operator on $L^2(\mathbb{K}^\star; L^2_g(\mathbb{T}^2))$, with domain $L^2(\mathbb{K}^\star; H^2(\mathbb{T}^2))$.

For any fixed $\theta \in \mathbb{R}^2$, $P^{\theta}_{g,\mathbb{T}^2}$ is the self-adjoint operator on $L^2_g(\mathbb{T}^2)$ defined by $P(D-A-\theta)+V$, (with domain $H^2(\mathbb{T}^2)$).

For any fixed $\Theta \in \mathbb{C}^2$, we will define in the same way the operator $P^{\Theta}_{g,\mathbb{T}^2}$, with the same domain. As $P^{\Theta}_{g,\mathbb{T}^2}$ is a perturbation of the Laplace–Beltrami operator, its spectrum is discrete (formed by eigenvalues with finite multiplicity and with no accumulation point).

Later on we will adopt the notation

$$P_{g,\mathbb{T}^2}(D-A-\Theta) = P^{\Theta}_{g,\mathbb{T}^2} - V. \tag{2.2}$$

$P_{g,\mathbb{T}^2}(D-A-\Theta)u(x) = P(D-A-\Theta)u(x), \forall u \in H^2(\mathbb{T}^2)$.

As for the case of the flat metric considered in [R-S] (see theorem 4.1.5 of [K] for general elliptic and periodic self-adjoint differential operator), the following theorem comes from Floquet theory.

Theorem 2.1. For any real open interval $(a,b) \subset \mathbb{R}, (a < b < +\infty)$, we have the equivalence

$$\begin{aligned}(a,b) \cap \mathrm{sp}(P_g(D-A)+V) &= (a,b) \cap \mathrm{sp}_{ac}((P_g(D-A)+V)\\ &\Longleftrightarrow (a,b) \cap \mathrm{sp}_p((P_g(D-A)+V) = \emptyset.\end{aligned} \tag{2.3}$$

Moreover

$$\begin{aligned}\mu \in \mathrm{sp}_p((P_g(D-A)+V) &\Longleftrightarrow \mu \in \mathrm{sp}_d(P_{g,\mathbb{T}^2}(D-A-\theta)+V) \qquad \forall \theta \in \mathbb{R}^2\\ &\Longleftrightarrow \mu \in \mathrm{sp}_d(P_{g,\mathbb{T}^2}(D-A-\Theta)+V) \qquad \forall \Theta \in \mathbb{C}^2.\end{aligned} \tag{2.4}$$

For an operator T : $\mathrm{sp}(T)$, $\mathrm{sp}_d(T)$, $\mathrm{sp}_p(T)$ and $\mathrm{sp}_{ac}(T)$ denote the spectrum, the discrete spectrum, the point spectrum (eigenvalues), and the absolutely continuous spectrum of T.

We recall that the spectrum of $P_g(D-A)+V$ is gauge invariant: the same is true for $P_{g,\mathbb{T}^2}(D-A)+V$,

$$\mathrm{sp}(P_g(D-A)+V) = \mathrm{sp}(P_g(D-A-\mathrm{d}\varphi)+V) \qquad \varphi(x) \in C^2(\mathbb{R}^2;\mathbb{R}) \tag{2.5}$$

and $\mathrm{sp}(P_{g,\mathbb{T}^2}(D-A)+V) = \mathrm{sp}(P_{g,\mathbb{T}^2}(D-A-\mathrm{d}\varphi)+V), \forall \varphi(x) \in C^2(\mathbb{T}^2;\mathbb{R})$.

We will use the corollary below.

Corollary 2.2. Let $\varphi(x) \in C^2(\mathbb{T}^2;\mathbb{R})$ and $\omega \in \mathbb{S}^1$ be given.

Then $\mu \in \mathrm{sp}_p((P_g(D-A)+V)$ iff

$$\mu \in \mathrm{sp}_d(P_{g,\mathbb{T}^2}(D-A-\theta-z(\mathrm{d}\varphi+\omega))+V) \qquad \forall (z,\theta) \in \mathbb{C}\times\mathbb{R}^2. \tag{2.6}$$

This comes from (2.4) when $\varphi = 0$ and the fact that $P_{g,\mathbb{T}^2}(D-A-\Theta)$ and $P_{g,\mathbb{T}^2}(D-A-\Theta-\mathrm{d}\psi)$ have the same eigenvalues, for any $\Theta \in \mathbb{C}$ and any $\psi \in C^2(\mathbb{T}^2;\mathbb{C})$.

3. Proof of theorem 1.1

The proof of theorem 1.1 comes from the assumption $V \in L^\infty(\mathbb{T}^2)$, (2.6) of corollary 2.2 and the theorem below.

Theorem 3.1. For any $\omega \in \mathbb{S}^1$, the unit sphere of $\mathbb{R}^2$, there exist $\theta_\omega \in \mathbb{R}^2$, $c_\omega > 0$, an integer $m = m(\omega) \in \mathbb{N}$ and a sequence of non-negative real numbers (λ_k), with $\lim_{k\mapsto\infty} \lambda_k = +\infty$, such that for any k,

$$\| P_{g,\mathbb{T}^2}(D-A-\theta_\omega - \mathrm{i}\lambda_k G(\omega))u \| \geqslant \lambda_k^{\frac{1}{m+1}} c_\omega \| u \| \qquad \forall u \in C^\infty(\mathbb{T}^2). \tag{3.1}$$

$$\begin{aligned}G(\omega) = \mathrm{d}(\omega x + \psi_\omega) \qquad &\text{with } \psi_\omega \in C^\infty(\mathbb{T}^2;\mathbb{R})\\ \int_{\mathbb{T}^2} \psi_\omega \,\mathrm{d}x = 0 \qquad &\text{s.t. } \Delta_g(\omega x + \psi_\omega) = 0.\end{aligned} \tag{3.2}$$

$\| \, . \, \|$ denotes the L^2_g-norm on $\mathbb{T}^2$, and $\Delta_g = -P_g(D)$ is the g-Laplace–Beltrami operator, and as $\Delta_g(\omega x)$ is always a periodic function orthogonal to the constants in $L^2_g(\mathbb{T}^2)$, the function ψ_ω exists and is unique.

Remark 3.2. We just have to prove equation (3.1) for the special conformal metric $|g|^{-1/2}g$.

If $\tilde{g} = |g|^{-1/2}g$, then $g^{jk}|g|^{1/2} = \tilde{g}^{jk}$ and $|\tilde{g}| = 1$.

The estimate (3.1) becomes

$$\int_{\mathbb{T}^2} |g|^{-1/2}|P_{\tilde{g},\mathbb{T}^2}(D - A - \theta_\omega - \mathrm{i}\lambda_k G(\omega))u|^2 \,\mathrm{d}x \geqslant \lambda_k^{\frac{2}{m+1}} c_\omega^2 \int_{\mathbb{T}^2} |g|^{1/2}|u|^2 \,\mathrm{d}x.$$

As $|g|$ and $|g|^{-1}$ are bounded, the estimate is equivalent to

$$\int_{\mathbb{T}^2} |P_{\tilde{g},\mathbb{T}^2}(D - A - \theta_\omega - \mathrm{i}\lambda_k G(\omega))u|^2 \,\mathrm{d}x \geqslant \lambda_k^{\frac{2}{m+1}} \tilde{c}_\omega^2 \int_{\mathbb{T}^2} |u|^2 \,\mathrm{d}x.$$

So from now on we will assume $|g| = 1$ and, due to (2.5) we will work in the particular gauge $\mathrm{div}(A^g) = 0$, where A^g is the vector field associated to the one-form A by the metric g, $(g(A^g, .) = A)$,

$$|g| = 1 \qquad A = J^\star(\mathrm{d}\phi) = -\sum_k g^{2k}\partial_{x_k}\phi(x)\,\mathrm{d}x_1 + \sum_k g^{1k}\partial_{x_k}\phi(x)\,\mathrm{d}x_2 \tag{3.3}$$

where $J^\star$ is the natural involution on $T^\star(\mathbb{T}^2)$ and $\phi(x)$ is the unique periodic function satisfying (we can choose a gauge A such that $\int_{\mathbb{T}^2} A^g \wedge \theta \,\mathrm{d}x = 0, \forall \theta \in \mathbb{R}^2$)

$$\Delta_g\phi(x) = B(x) \qquad \text{and} \qquad \int_{\mathbb{T}^2} \phi(x)\,\mathrm{d}x = 0. \tag{3.4}$$

For the proof of theorem 3.1, let us introduce the Pauli operators on $(L^2(\mathbb{T}^2))^2$.

For any complex one-form $N = \sum_j N_j(x)\,\mathrm{d}x_j$, $N_j \in C^\infty(\mathbb{T}^2; \mathbb{C})$, we let

$$M^\mp(D - N) = \sum_k h^{1k}(D_{x_k} - N_k(x)) \mp \mathrm{i}\sum_k h^{2k}(D_{x_k} - N_k(x)) \tag{3.5}$$

and

$$\tilde{M}^\mp(D - N) = \sum_k (D_{x_k} - N_k(x))h^{1k} \mp \mathrm{i}\sum_k (D_{x_k} - N_k(x))h^{2k}. \tag{3.6}$$

The matrix (h^{jk}) is the square root of (g^{jk}):

$$h^{jj} > 0 \qquad h^{jk} = h^{kj} \qquad \sum_q h^{jq}h^{qk} = g^{jk}.$$

$((M^\mp(D - N))^\star = \tilde{M}^\pm(D - \bar{N}).)$

With the choice of A in (3.3) we get, $\forall\Theta \in \mathbb{C}^2$,

$$\begin{aligned} &\mathrm{e}^{\mp\phi}M^\mp(D - A - \Theta)\mathrm{e}^{\pm\phi} = M^\mp(D - \Theta) \\ &\mathrm{e}^{\mp\phi}\tilde{M}^\mp(D - A - \Theta)\mathrm{e}^{\pm\phi} = \tilde{M}^\mp(D - \Theta) \end{aligned} \tag{3.7}$$

(on the basis that, if h is the matrix $h = (h_{jk})$ and $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ the Pauli matrix, then $h^{-1}\sigma_2 = \sigma_2 h$).

Let us define the two Pauli operators:

$$\mathcal{D}(D - N) = \begin{pmatrix} 0 & M^-(D - N) \\ M^+(D - N) & 0 \end{pmatrix} \tag{3.8}$$

$$\tilde{\mathcal{D}}(D - N) = \begin{pmatrix} 0 & \tilde{M}^-(D - N) \\ \tilde{M}^+(D - N) & 0 \end{pmatrix}. \tag{3.9}$$

If $\mathrm{d}N = B_N\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$, then

$$\tilde{\mathcal{D}}(D-N)\mathcal{D}(D-N) = \begin{pmatrix} P(D-N) - B_N & 0 \\ 0 & P(D-N) + B_N \end{pmatrix} \tag{3.10}$$

and $\tilde{\mathcal{D}}(D-\bar{N}) = (\mathcal{D}(D-N))^\star$.

Lemma 3.3. If $\mu_1(\lambda,\omega,\theta)$ is the first eigenvalue of $P_{g,\mathbb{T}^2}(D-A-\theta) + \lambda^2|G(\omega)|^2_{g_x}$, then

$$\| \mathcal{D}(D-A-\theta-\mathrm{i}\lambda G(\omega))\mathcal{I}(u) \| \geqslant (2\mu_1(\lambda,\omega,\theta))^{1/2} \| u \|, \forall u \in C^1(\mathbb{T}^2) \tag{3.11}$$

and $\forall(\theta,\omega,\lambda) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$, if $\mathcal{I}(u) = \begin{pmatrix} u \\ u \end{pmatrix}$, for any $u \in L^2(\mathbb{T}^2)$.

Proof of lemma 3.3. Taking into account that $|g| = 1$ by (3.3), we check the identities

$$\begin{aligned} M^{\mp}(D-A-\theta-\mathrm{i}\lambda G(\omega)) &= M^{\mp}(D-A-\theta \pm \lambda R(\omega)) \\ \tilde{M}^{\mp}(D-A-\theta-\mathrm{i}\lambda G(\omega)) &= \tilde{M}^{\mp}(D-A-\theta \pm \lambda R(\omega)) \end{aligned} \tag{3.12}$$

with $R(\omega)$ the real closed one-form (thanks to (3.2)), linear in ω, defined by

$$R(\omega) = J^\star(G(\omega)) \qquad (\mathrm{d}R(\omega) = \Delta_g(\omega x + \psi_\omega) = 0). \tag{3.13}$$

So, as for (3.10) we get

$$\begin{aligned} &(\mathcal{D}(D-A-\theta-\mathrm{i}\lambda G(\omega)))^\star \mathcal{D}(D-A-\theta-\mathrm{i}\lambda G(\omega)) \\ &\quad = \begin{pmatrix} P_{g,\mathbb{T}^2}(D-A-\theta-\lambda R(\omega)) - B & 0 \\ 0 & P_{g,\mathbb{T}^2}(D-A-\theta+\lambda R(\omega)) + B \end{pmatrix} \end{aligned} \tag{3.14}$$

and then, for any function $u \in C^2(\mathbb{T}^2)$,

$$\begin{aligned} &\| \mathcal{D}(D-A-\theta-\mathrm{i}\lambda G(\omega))\mathcal{I}(u) \|^2 \\ &\quad = 2 < P_{g,\mathbb{T}^2}(D-A-\theta)u|u >_{L^2(\mathbb{T}^2)} + 2\lambda^2 \int_{\mathbb{T}^2} |G(\omega)|^2_g \times |u|^2\,\mathrm{d}x. \end{aligned} \tag{3.15}$$

We used the fact that $J^\star$ is isometric, $|R(\omega)|_g = |J^\star(G(\omega))|_g = |G(\omega)|_g$. □

Lemma 3.4. Let $\omega \in \mathbb{S}^1$. There exists $\theta^0 = \theta^0(\omega) \in \mathbb{R}^2$ satisfying: for any $\eta > 0$ there exists $c_2(\eta) = c_2(\eta,\omega) > 0$, such that, $\forall\theta \in \mathbb{R}^2$ such that $d_0(\theta \pm (\theta^0 + \lambda\theta^1(\omega)); \Gamma^\star) \geqslant \eta$, then

$$\| \tilde{\mathcal{D}}(D-A-\theta-\mathrm{i}\lambda G(\omega))U \| \geqslant c_2(\eta) \| U \| \qquad \forall U \in (C^1(\mathbb{T}^2))^2 \qquad \forall\lambda \in \mathbb{R} \tag{3.16}$$

$d_0(\cdot;\cdot)$ denotes the standard Euclidean distance and

$$\theta^1(\omega) = (\theta_1(\omega), \theta_2(\omega)) = \left(-\sum_j g_0^{2j} w_j, \sum_j g_0^{1j} w_j \right)$$

$$g_0^{kj} = \frac{1}{|\mathbb{T}^2|} \int_{\mathbb{T}^2} \left[g_x^{kj} + \left(\sum_m \partial_{x_m} g_x^{km} \right) (\Delta_g)^{-1} \left(\sum_m \partial_{x_m} g_x^{jm} \right) \right] \mathrm{d}x.$$

Proof of lemma 3.4. Let $R(\omega)$ be the one-form defined by (3.13) and let

$$N_g = \sum_{j,k} \partial_{x_j} h^{kj} (-h^{2k}\,\mathrm{d}x_1 + h^{1k}\,\mathrm{d}x_2). \tag{3.17}$$

Then, if $U = \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \in (C^1(\mathbb{T}^2))^2$,

$$\| \tilde{\mathcal{D}}(D - A - \theta - \mathrm{i}\lambda G(\omega))U \|^2 = \| M^+(D - A - N_g - \lambda R(\omega) - \theta)u^+ \|^2 + \| M^-(D - A + N_g + \lambda R(\omega) - \theta)u^- \|^2. \tag{3.18}$$

But, as for the definition of $\phi(x)$ in (3.3) and (3.4), we can find $\theta^0 \in \mathbb{R}^2$, two real, periodic functions $\psi(x)$ and $\varphi(x)$ such that

$$N_g = \mathrm{d}(\theta^0 x) + \mathrm{d}\varphi(x) + J^\star(\mathrm{d}\psi). \tag{3.19}$$

$(\psi, \varphi \in C^\infty(\mathbb{T}^2; \mathbb{R}).)$ By (3.13),

$$R(\omega) = \mathrm{d}(\theta^1(\omega)x + \varphi_\omega) \qquad \text{with } \varphi_\omega(x) \in C^\infty(\mathbb{T}^2; \mathbb{R}). \tag{3.20}$$

Taking into account (3.7) and (3.18), we get

$$\begin{aligned} \| \tilde{\mathcal{D}}(D - A - \theta - \mathrm{i}\lambda G(\omega))U \|^2 = {} & \| \mathrm{e}^{-(\phi+\psi-\mathrm{i}\varphi-\mathrm{i}\lambda\varphi_\omega)} M^+(D - \theta^0 - \lambda\theta^1(\omega) - \theta) \\ & \times \mathrm{e}^{(\phi+\psi-\mathrm{i}\varphi-\mathrm{i}\lambda\varphi_\omega)} u^+ \|^2 + \| \mathrm{e}^{-(-\phi+\psi+\mathrm{i}\varphi+\mathrm{i}\lambda\varphi_\omega)} \\ & \times M^-(D + \theta^0 + \lambda\theta^1(\omega) - \theta) \mathrm{e}^{(-\phi+\psi+\mathrm{i}\varphi+\mathrm{i}\lambda\varphi_\omega)} u^- \|^2. \end{aligned} \tag{3.21}$$

So there exists a constant $C_{A,g} > 0$ such that

$$\begin{aligned} & \| \tilde{\mathcal{D}}(D - A - \theta - \mathrm{i}\lambda G(\omega))U \|^2 \\ & \quad \geqslant C_{A,g}\{\| M^+(D - \theta^0 - \lambda\theta^1(\omega) - \theta)\mathrm{e}^{(\phi+\psi-\mathrm{i}\varphi-\mathrm{i}\lambda\varphi_\omega)} u^+ \|^2 \\ & \qquad + \| M^-(D + \theta^0 + \lambda\theta^1(\omega) - \theta)\mathrm{e}^{(-\phi+\psi+\mathrm{i}\varphi+\mathrm{i}\lambda\varphi_\omega)} u^- \|^2\} \\ & \quad = C_{A,g}\{\langle P_{g,\mathbb{T}^2}(D - \theta^0 - \lambda\theta^1(\omega) - \theta)\mathrm{e}^{(\phi+\psi-\mathrm{i}\varphi-\mathrm{i}\lambda\varphi_\omega)} u^+ | \mathrm{e}^{(\phi+\psi-\mathrm{i}\varphi-\mathrm{i}\lambda\varphi_\omega)} u^+ \rangle_{L^2(\mathbb{T}^2)} \\ & \qquad + \langle P_{g,\mathbb{T}^2}(D + \theta^0 + \lambda\theta^1(\omega) - \theta)\mathrm{e}^{(-\phi+\psi+\mathrm{i}\varphi+\mathrm{i}\lambda\varphi_\omega)} u^- | \mathrm{e}^{(-\phi+\psi+\mathrm{i}\varphi+\mathrm{i}\lambda\varphi_\omega)} u^- \rangle_{L^2(\mathbb{T}^2)}\}. \end{aligned} \tag{3.22}$$

Finally, changing $C_{A,g}$ from (3.22) the estimate follows

$$\begin{aligned} & \| \tilde{\mathcal{D}}(D - A - \theta - \mathrm{i}\lambda G(\omega))U \|^2 \\ & \quad \geqslant C_{A,g}\{\| (D - \theta^0 - \lambda\theta^1(\mathrm{theta}^1(\omega) - \theta)\mathrm{e}^{(\phi+\psi-\mathrm{i}\varphi-\mathrm{i}\lambda\varphi_\omega)} u^+ \|^2 \\ & \qquad + \| (D + \theta^0 + \lambda\theta^1(\omega) - \theta)\mathrm{e}^{(-\phi+\psi+\mathrm{i}\varphi+\mathrm{i}\lambda\varphi_\omega)} u^- \|^2\} \end{aligned} \tag{3.23}$$

which proves (3.16). □

Lemma 3.5. For any $\omega \in \mathbb{S}^1$, there exists an integer $m \in \mathbb{N}$ and a constant C_ω such that

$$\mu_1(\lambda, \omega, \theta) \geqslant \lambda^{\frac{2}{m+1}}/C_\omega \qquad \forall \lambda > 1/C_\omega \qquad \theta \in \mathbb{K}. \tag{3.24}$$

$\mu_1(\lambda, \omega, \theta)$ denotes the first eigenvalue of $P_{g,\mathbb{T}^2}(D - A - \theta) + \lambda^2 |G(\omega)|^2_{g_x}$.

Proof of lemma 3.5. The min–max principle (see [R-S]), gives the formula of the ground state energy

$$\mu_1(\lambda, \omega, \theta) = \inf_{\|u\|=1} \left\{ \| M^+(D - A - \theta)u \|^2 + \int_{\mathbb{T}^2} (\lambda^2 |G(\omega)|^2_{g_x} + B(x))|u|^2 \,\mathrm{d}x \right\}. \tag{3.25}$$

From (3.25), (3.18) and (3.21) it is sufficient to prove (3.24) when $A = 0$, (B is bounded). If $|G(\omega)|_{g_x} > 0, \forall x \in \mathbb{T}^2$, then (3.24) is obvious with $m = 0$.

But the zeros of the function $x \mapsto |G(\omega)|_{g_x}$, if they exist, are isolated:

$$|G(\omega)|_{g_x} = 0 \Longleftrightarrow x \in \mathcal{Z} = \{z_1, \ldots, z_N\}. \tag{3.26}$$

More precisely, for each zero z_j there exists an integer $m_j \in \mathbb{N}$ and a constant C_j such that

$$C_j^{-1}|x-z_j|^{m_j} \leqslant |G(\omega)|_{g_x} \leqslant C_j|x-z_j|^{m_j} \qquad \text{if } |x-z_j|^{m_j} \leqslant C_j^{-1}. \tag{3.27}$$

To be convinced, recall that $G(\omega) = \mathrm{d}f_\omega$ with $f_\omega(x) = \omega x + \psi_\omega(x)$ which is a g-harmonic function. But it is well known that for any $x_0 \in \mathbb{T}^2$, there exist local coordinates in a neighbourhood of x_0 such that the metric g becomes conformal to the flat one, see for example [Sp]. (Take for example $y = (y_1(x), y_2(x))$ with $y_j(x) g$-harmonic functions, $\mathrm{d}y_{1,x_0} \neq 0$ and $\mathrm{d}y_2 = J^\star(\mathrm{d}y_1)$.)

So the function f_ω is (locally) the real part of some holomorphic function, for some locally complex structure, and then (3.26) and (3.27) are valid.

We get from (3.26) and (3.27) that, if $A = 0$, there exists C_ω such that

$$\inf_j\{\mu_{1,j}(\lambda,\omega)\} \leqslant \inf_j\{\mu^0_{1,j}(\lambda,\omega)\} \leqslant C^0_\omega \mu_1(\lambda,\omega,\theta) \qquad \forall\lambda > C_\omega \qquad \theta \in \mathbb{K} \tag{3.28}$$

for some constant C^0_ω, if $\mu_{1,j}(\lambda,\omega)$ is the first eigenvalue of the Schrödinger operator $-\Delta + \lambda^2 C_\omega^{-1}|x-z_j|^{2m_j}$ on $L^2(\mathbb{R}^2)$, and if $\mu^0_{1,j}(\lambda,\omega)$ is the one for the Dirichlet problem on the ball $B(z_j; C_\omega^{-1})$ of the same operator.

So by scaling we get from (3.28)

$$\inf_j\{\lambda^{\frac{2}{m_j+1}}\mu_{1,j}\} \leqslant C_\omega \mu_1(\lambda,\omega,\theta) \qquad \forall\lambda > C_\omega \qquad \theta \in \mathbb{K} \tag{3.29}$$

if $\mu_{1,j}$ is the first eigenvalue of the Schrödinger operator $-\Delta + |x|^{2m_j}$ on $L^2(\mathbb{R}^2)$.

The second estimate of (3.28) (the right one), can be obtained in the following way.

We take a smooth partition of unity $\{\varphi_0, \varphi_1, \ldots, \varphi_N\}$ on $\mathbb{T}^2$ such that

$$\sum_{j=0}^N \varphi_j^2(x) = 1 \qquad \text{Supp }(\varphi_j) \subset B(z_j; C_\omega^{-1}) \qquad \text{Supp }(\varphi_0) \cap B(z_j; \tfrac{1}{2}C_\omega^{-1}) = \emptyset$$
$$\text{for } j = 1, \ldots, N.$$

For any $u \in H^1(\mathbb{T}^2)$, since $M^+(D)$ is a one-form

$$\sum_{j=0}^N \varphi_j M^+(D)(\varphi_j) = 0 \qquad \text{and} \qquad M^+(D)(\varphi_j u) = \varphi_j M^+(D)(u) + u M^+(D)(\varphi_j)$$

it follows that

$$\begin{aligned}\| M^+(D)u \|^2 &= \sum_{j=0}^N \| \varphi_j M^+(D)u \|^2 = \sum_{j=0}^N [\| M^+(D)(\varphi_j u) - u M^+(D)(\varphi_j) \|^2 \\ &= \sum_{j=0}^N [\| M^+(D)(\varphi_j u) \|^2 + \| u M^+(D)(\varphi_j) \|^2] \\ &\quad - \sum_{j=0}^N [\langle M^+(D)(\varphi_j u) | u M^+(D)(\varphi_j)\rangle + \langle u M^+(D)(\varphi_j) | M^+(D)(\varphi_j u)\rangle].\end{aligned}$$

Writing

$$\begin{aligned}\langle M^+(D)(\varphi_j u) | u M^+(D)(\varphi_j)\rangle + \langle u M^+(D)(\varphi_j) | M^+(D)(\varphi_j u)\rangle = 2 \| u M^+(D)(\varphi_j) \|^2 \\ + \langle M^+(D)(u) | u\varphi_j M^+(D)(\varphi_j)\rangle + \langle u\varphi_j M^+(D)(\varphi_j) | M^+(D)(u)\rangle\end{aligned}$$

then summing, we get the equality

$$\| M^+(D)u \|^2 = \sum_{j=0}^N \| M^+(D)(\varphi_j u) \|^2 - \int_{\mathbb{T}^2} |u|^2 \sum_{j=0}^N |M^+(D)(\varphi_j)|^2 \,\mathrm{d}x.$$

We recall that $\mu^0_{1,j}(\lambda,\omega) \leqslant \lambda^2 C_\omega^{-2m_j-1} + \mu^0_1(\omega) \leqslant \lambda^2 C'_\omega$, for some constant $C'_\omega > 1$, if $\lambda > C_\omega$, ($\mu^0_1(\omega)$ is the first eigenvalue of the Dirichlet problem associated to $-\Delta$ on $B(z_j; C_\omega^{-1})$).

We recall also that

$$\| M^+(D)(\varphi_0 u) \|^2 + \lambda^2 \| |G(\omega)|_g \varphi_0 u \|^2 \geqslant \lambda^2 (C''_\omega)^{-1} \| \varphi_0 u \|^2$$
$$\geqslant (C^1_\omega)^{-1} \inf_k \mu^0_{1,k}(\lambda,\omega) \| \varphi_0 u \|^2$$

for some constant $C''_\omega > 1$, if $C^1_\omega = C'_\omega C''_\omega$.

So, for any $j = 0, \ldots, N$,

$$\| M^+(D)(\varphi_j u) \|^2 + \lambda^2 \| |G(\omega)|_g \varphi_j u \|^2 \geqslant (C^1_\omega)^{-1} \inf_k \mu^0_{1,k}(\lambda,\omega) \| \varphi_j u \|^2$$

and then

$$\| M^+(D)(u) \|^2 + \lambda^2 \| |G(\omega)|_g u \|^2 \geqslant [(C^1_\omega)^{-1} \inf_k \mu^0_{1,k}(\lambda,\omega) - C_{\omega,2}] \| \varphi_j u \|^2$$

with $C_{\omega,2}$ the maximum on $\mathbb{T}^2$ of $\sum_j |M^+(D)(\varphi_j)|^2$.

The second estimate of (3.28) follows for large λ, ($\mu^0_{1,j}(\lambda,\omega) \mapsto +\infty$ when $\lambda \mapsto +\infty$). □

End of the proof of theorem 3.1. We have seen that it is sufficient to take $\Gamma = \mathbb{Z}^2$. Let θ_ω such that $\theta_{\omega,1} \pm \theta^0_1 \notin 2\pi\mathbb{Z}$ and let $\mu > 0$ such that $\mu\theta^1_1(\omega) \in 2\pi\mathbb{Z}$.

Then we can take $(\lambda_k) = (n(k)\mu)$ for any increasing sequence of non-negative integer $(n(k))$, therefore there exists $\eta > 0$ such that $d_0(\theta_\omega \mp (\theta^0 + \lambda_k \theta^1(\omega)); \Gamma^\star) > \eta$.

With $\lambda = \lambda_k$, applying lemma 3.4 to $U = \mathcal{D}(D - A - \theta_\omega - \mathrm{i}\lambda_k G(\omega))\mathcal{I}(u)$ and then using lemma 3.3 to estimate U above, we get (3.1) from (3.10), (3.16), (3.11), lemma 3.5 and from $B \in L^\infty(\mathbb{T}^2)$. □

Remark 3.6. There exists $\omega \in \mathbb{S}^1$ such that $m = 0$ in (3.1), iff $F^\star(g) = c(y)g_0$, for some non-negative and $u(\Gamma)$–periodic function $c(y) \in C^\infty(\mathbb{R}^2)$, with $u \in GL(2;\mathbb{R})$ and F is a $\mathbb{R}^2$-diffeomorphism of the form $F(x) = u(x) + (\psi_1(x), \psi_2(x))$, with $\psi_j(x) \in C^\infty(\mathbb{T}^2)$.

If $\omega \in \mathbb{S}^1$ is such that $|G(\omega)|_{g_x} > 0$, then $\psi_1(x) = \psi_\omega(x) - \psi_\omega(0)$ and $F = (f_1, f_2)$, $f_1(x) = \omega x + \psi_1(x)$, and $f_2(x) = \tilde{\omega} x + \psi_2(x)$ is the g-harmonic function such that $\mathrm{d} f_2 = J^\star(\mathrm{d} f_1)$ and $f_2(0) = 0$.

The f_j are g-harmonic and $F(\Gamma) = u(\Gamma)$, $u(x) = (\omega x, \tilde{\omega} x)$.

Otherwise, using semiclassical analysis method as in [H-M-1], one can get easily the equivalence in (3.24): $\mu_1(\lambda,\omega,\theta)/\lambda^{\frac{2}{m+1}}$ is also bounded.

Acknowledgments

We are grateful to the referees whose remarks were useful for perfecting the proof.

References

[B-S] Birman M Sh and Suslina T A 1998 Two-dimensional periodic magnetic hamiltonian is absolutely continuous *St. Petersburg Math. J.* **9** 21–32

[G] Green E L 1997 Spectral theory of Laplace–Beltrami operators with periodic metrics *J. Diff. Eqns* **133** 15–29

[H-M-1] Helffer B and Mohamed A 1996 Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells *J. Funct. Anal.* **138** 40–81

[K] Kuchment P 1993 Floquet theory for partial differential equations *Operator Theory, Advances and Applications* vol 60 (Basel: Birkhäuser)

[Mh] Mohamed A 1997 Asymptotic of the density of states for the Schrodinger operator with electromagnetic potential *J. Math. Phys.* **38** 4023–51

[R-S] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics IV* (New York: Academic)

[Sk] Skriganov M M 1979 Proof of the Bethe–Sommerfeld conjecture in dimension two *Sov. Math. Dokl.* **20** 956–9

[So] Sobolev A 1997 Absolute continuity of the periodic magnetic Schrödinger operator *Mathematical Physics Preprint* mp-arc, 97-637

[Sp] Spivak M 1975 *Differential Geometry* (Houston, TX: Publish or Perish)

[T] Thomas L E 1973 Time dependent approach to scattering from impurities in a cristal *Commun. Math. Phys.* **33** 335–43